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# BOUNDARIES AND GENERATION OF CONVEX SETS

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ABSTRACT

We introduce a notion which is intermediate between that of taking the $w^*$-closed convex hull of a set and taking the norm closed convex hull of this set. This notion helps to streamline the proof (given in [FLP]) of the famous result of James in the separable case. More importantly, it leads to stronger results in the same direction. For example:

1. Assume $X$ is separable and non-reflexive and its unit sphere is covered by a sequence of balls $\{C_i\}_{i=1}^{\infty}$ of radius $a < 1$. Then for every sequence of positive numbers $\{\varepsilon_i\}_{i=1}^{\infty}$ tending to 0 there is an $f \in X^*$, such that $\|f\| = 1$ and $f(x) \leq 1 - \varepsilon_i$, whenever $x \in C_i$, $i = 1, 2, \ldots$.

2. Assume $X$ is separable and non-reflexive and let $T\colon Y \to X^*$ be a bounded linear non-surjective operator. Then there is an $f \in X^*$ which does not attain its norm on $B_X$ such that $f \notin T(Y)$.

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## 1. Introduction

Let $X$ be a real Banach space and let $K$ be a $w^*$-compact convex subset of $X^*$. A subset $B \subset K$ is called a boundary for $K$ if for every $x \in X$ there is an $f \in B$ so that $f(x) = \sup\{g(x) : g \in K\}$. Known results in the literature show that under certain assumptions (if $B$ is norm-separable [R] or if $X$ is separable and $l_1 \not\subset X$, i.e., $X$ does not have a subspace isomorphic to $l_1$, [G]) the set $K$ is the norm closed convex hull of $B$. The result of [R] mentioned above implies the famous result of James [J] in the separable case. Indeed, let $B$ be a closed convex bounded and separable subset of the separable Banach space $X$ so that every $f \in X^*$ attains its maximum on $B$. The $w^*$-closure $K$ of $B$ in $X^{**}$ is a $w^*$-compact convex set. By the assumption $B$ is a boundary for $K$. Hence, by the result just quoted, the norm-closed convex hull of $B$ (which is $B$ itself) coincides with $K$ and thus $B$ is weak-compact.

In [FLP] a new proof is given to the result of [R]. By analysing this proof we were led to a new concept of generating a $w^*$-compact convex set $K$ by a subset. This concept is intermediate between taking the $w^*$-closed convex hull and the norm-closed convex hull. In Section 2 we formally define this intermediate notion and apply it to derive the results of [R] and [G] mentioned above. The proof of these results is essentially the same as in [FLP] but the formulation is somewhat different.

In Sections 3 and 4 we demonstrate that this new concept is of interest also in some other contexts and use it to prove some new results. In Section 3 we take a (usually non-closed) cone $A$ in a separable space $X$ with vertex at the origin (i.e., $x \in A \Longrightarrow \lambda x \in A$ for all real $\lambda$). We consider a subset $B \subset S_{X^*}$ such that for any $x \in A$ with $||x|| = 1$ there is an $f \in B$ with $f(x) = 1$. The question we consider is "how massive has $A$ to be in order that $B_{X^*}$ is the norm-closed convex hull of $B$?" (we assume of course as before that either $B$ is separable or $X \not\supset l_1$). By using the method of proof in Section 2 we are able to give a quite satisfactory answer to this question. It turns out that to get this result we have to work not only with the given norm in $X$ but with the set of all equivalent norms.

We also prove in Section 3 that if $X \not\supset l_1$, then for $A$ to be massive enough for our purpose it suffices that the complement of $A$ can be covered by a proper operator range (i.e., there is a Banach space $Y$ and a bounded linear non-surjective operator $T$: $Y \to X$ so that $X \setminus A \subset T(Y)$). In particular, any cone $A$ whose complement is generated by a countable set will do.

In Section 4 we study, for separable non-reflexive $X$, the structure of the continuous linear functionals on $X$ which do not attain their norm. This topic has been studied quite a lot in the literature. We mention here just three results which are relevant to our discusion in this paper.

(i) The theorem of Bishop and Phelps [BP] which states that for every closed convex bounded $K \subset X$ the functionals which attain their maximum on $K$ form a norm-dense set in $X^*$.

(ii) The theorem of Phelps [P] which states that if $X$ has the Radon–Nikodym property (RNP) the functionals which attain their norm on the unit ball form a dense $G_\delta$-set in $X^*$ and thus those which do not attain their norm form a set of the first category.

(iii) The result of Bourgain and Stegall (see [B], Theorem 3.3.5). This result yields that if $K$ is a non-dentable closed convex and bounded set in a separable Banach $X$ space, then the set of all support functionals of $K$ forms a set of the first category in $X^*$.

Other papers which have some relevant results are [NR] and [JM].

One of the main results proved in Section 4 is that if $X$ is separable and non-reflexive and if we cover $B_X$ by a sequence of balls $\{D_n\}_{n=1}^\infty$ of radius $a < 1$, then for any sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \to 0$ as slowly as we wish there is an $f \in X^*$ of norm 1 so that $\sup\{f(x) : x \in D_n\}$ is at most $1 - \varepsilon_n$ for every $n$. This is proved by the approach developed in Section 2.

Another result in Section 4 is that the set of functionals in $X^*$ which do not attain their norm on $B_X$ can never be covered by a proper operator range. Proper operator ranges are sets of the first category of a special type. Thus it is of interest to compare this result with the result (ii) mentioned above.

For background to this paper and for notation we refer to the survey [FLP]. We just recall that if $K$ is any set in a Banach space $X$, we denote $\Sigma(K)$ the set of non-zero functionals in $X^*$ which attain their maximum on $K$.

## 2. Intermediate representation property

Let $B$ be a subset of a $w^*$-compact convex set $K$ in the dual $X^*$ of a Banach space $X$. Besides the usual ways in which $B$ can generate $K$, namely (W) ($K$ is the $w^*$-closed convex hull of $B$) and (S) ($K$ is the norm-closed convex hull of $B$), we introduce now an intermediate way (I).

*Definition 2.1:* We say that $B$ (I)-generates $K$ if for every representation of $B$ as $\bigcup_{n=1}^\infty C_n$ we have that $K$ is the norm-closed convex hull of $\bigcup_{n=1}^\infty w^* \operatorname{cl} \operatorname{co} C_n$. Equivalently, if $B$ is the union of an increasing sequence of sets $\{C_n\}_{n=1}^\infty$ then $\bigcup_{n=1}^\infty w^* \operatorname{cl} \operatorname{co} C_n$ is norm-dense in $K$.

It is clear that (S)$\Longrightarrow$(I)$\Longrightarrow$(W). In general, none of these arrows can be inverted. If, for example, $X = C[0,1]$ and $K = B_{X^*}$, let $B_1 = \pm Q$ (the rational points considered in a natural way as a subset of $K$) and $B_2 = \pm[0,1]$; then $B_1$ (W)-generate $K$, but clearly does not (I)-generate $K$. Also, $B_2$ (I)-generates $K$ (see Theorem 2.3 below) but does not (S)-generate $K$.

In some important special cases (I) and (S) coincide.

PROPOSITION 2.2: *Let $B$ be a subset of a $w^*$-compact set $K$ in $X^*$. Assume that either one of the following hold:*

(a) *$B$ is norm-separable.*

(b) *$X$ is separable and does not contain a subspace isomorphic to $l_1$.*

*Then if $B$ (I)-generates $K$ it also (S)-generates $K$.*

*Proof:* Assume that (a) holds, and let $\{h_n\}_{n=1}^\infty$ be a dense sequence in $B$. Let $\varepsilon > 0$ and put $C_n = (h_n + \varepsilon B_{X^*}) \cap B$. Then if $B$ (I)-generates $K$ then $K = \mathrm{cl\,co} \bigcup_{n=1}^\infty w^* \mathrm{cl\,co}\, C_n$. Since $\varepsilon > 0$ is arbitrary it follows that $K = \mathrm{cl\,co}\, B$.

Assume now that (b) holds and that $B$ (I)-generates $K$ but $K \neq \mathrm{cl\,co}\, B$. Then by the separation theorem there are an $F \in X^{**}$ and a constant $a$ so that

$$\sup\{F(g) : g \in K\} > a > \sup\{F(g) : g \in B\}.$$

By the result of Odell and Rosenthal [OR] there is a sequence $\{x_n\}_{n=1}^\infty$ in $X$ so that $F = w^* \lim_n x_n$. Put $C_n = \{g \in K : g(x_i) \leq a, i > n\}$. Clearly each $C_n$ is $w^*$-compact, $\bigcup_{n=1}^\infty C_n \supset B$, and $\sup\{F(g) : g \in C_n\} \leq a$ for every $n$. Since $B$ (I)-generates $K$ it follows that $K = \mathrm{cl\,co} \bigcup_{n=1}^\infty C_n$ and, in particular, $F(g) \leq a$ for every $g \in K$, a contradiction. ∎

The main interest in (I) stems from the following result.

THEOREM 2.3: *Let $K$ be a $w^*$-compact convex subset of $X^*$ and $B \subset K$ be a boundary. Then $B$ (I)-generates $K$.*

*Proof:* Without loss of generality we may assume that $0 \in B$. Let $B = \bigcup_{i=1}^\infty C_i$. Fix an $\varepsilon > 0$ and a sequence of positive numbers $\{\varepsilon_i\}_{i=0}^\infty$ so that $\sum_{i=0}^\infty \varepsilon_i < \varepsilon$. Put

$$C_0 = \varepsilon_0 B_{X^*}, \quad K_1 = \mathrm{co}(K \cup C_0), \quad B_1 = \bigcup_{i=0}^\infty (1+\varepsilon_i) C_i, \quad V^* = w^* \mathrm{cl\,co}\, B_1.$$

Clearly $B \subset \mathrm{co}\, B_1$ and, since $B$ is a boundary, it follows from the separation theorem that $K_1 \subset V^*$. It is also clear that $V^*$ is a $w^*$-closed body in $X^*$ with $0 \in \mathrm{int}\, V^*$. Let $V$ be the polar of $V^*$ in $X$.

By the Krein–Milman theorem there is, for every $f \in V^*$, a probability measure $\mu$ on the $w^*$-closure of $B_1$ representing $f$, i.e.,

$$f(x) = \int_{w^* \operatorname{cl} B_1} g(x) d\mu(g), \quad x \in X.$$

Let $f \in \Sigma(V)$ and $x \in \partial V$ be such that $f(x) = 1$. It is clear that $\operatorname{supp} \mu \subset w^* \operatorname{cl} B_1 \cap \{g \in V^* : g(x) = 1\}$. We claim that this set is in turn contained in

$$\bigcup_{i=0}^{l} w^* \operatorname{cl}(1+\varepsilon_i) C_i$$

for some finite $l$. Indeed, if that were not the case then, by using $\lim_i \varepsilon_i = 0$, we would get $\{g \in V^* : g(x) = 1\} \cap K \neq \emptyset$ and thus $\max\{g(x) : g \in K\} = \max\{g(x) : g \in V^*\} = 1$. Since $B = \bigcup_{i=1}^{\infty} C_i$ is a boundary of $K$, it follows that for some $j$ there is an $h \in C_j$, with $h(x) = 1$. However, since $(1+\varepsilon_j)h_j \in V^*$ and $(1+\varepsilon_j)h_j(x) = 1 + \varepsilon_j > 1$ we have arrived at a contradiction, and our claim is proved.

Put $D_0 = (1+\varepsilon_0)C_0$ and, for $n \geq 1$,

$$D_n = w^* \operatorname{cl}(1+\varepsilon_n)C_n \setminus \bigcup_{i=0}^{n-1} w^* \operatorname{cl}(1+\varepsilon_n)C_i.$$

Let $\sigma = \{n : \mu_n = \mu(D_n) > 0\}$ and let $h_n$ be the barycenter of $\mu_n^{-1}\mu$ restricted to $D_n$, $n \in \sigma$. Clearly, $h_n \in w^* \operatorname{cl} \operatorname{co}(1+\varepsilon_n)C_n$ for every $n \in \sigma$. It is also clear that $f = \sum_{n \in \sigma} \mu_n h_n$. Put

$$g = \sum_{n \in \sigma, n > 0} \mu_n (1+\varepsilon_n)^{-1} h_n.$$

Since $0 \in B$ (and hence to some $C_n, n > 0$) it follows that

$$g \in \operatorname{cl} \operatorname{co} \bigcup_{i=1}^{\infty} w^* \operatorname{cl} \operatorname{co} C_i.$$

Note that $||f - g|| \leq \varepsilon(1 + L)$ where $L = \sup\{||t|| : t \in K\}$. Consequently, by the Bishop–Phelps theorem we get that for every $f \in \partial V^*$ there is a $g \in \operatorname{cl} \operatorname{co} \bigcup_{i=1}^{\infty} w^* \operatorname{cl} \operatorname{co} C_i$ with $||f - g|| \leq \varepsilon(L+2)$. The same is true for every $f \in V^*$ and, in particular, for $f \in K$. Letting $\varepsilon \to 0$ concludes the proof. ∎

*Remark:* That $\operatorname{supp}\mu \subset \bigcup_{i=0}^{l} w^* \operatorname{cl}(1+\varepsilon_i)C_i$ for some finite $l$ was not really used in the proof above. What is essential is only the fact that

$$\operatorname{supp}\mu \subset \bigcup_{i=0}^{\infty} w^* \operatorname{cl}(1+\varepsilon_i)C_i.$$

By combining Proposition 2.2 with Theorem 2.3 we get a new proof to the following known corollary.

COROLLARY 2.4 ([R], [G]): *Let $B$ be a boundary of a $w^*$-compact convex $K$ in $X^*$. Assume either of the following:*

(a) *$B$ is norm-separable.*

(b) *$X$ is separable and $X \not\supset l_1$.*

*Then $B$ (S)-generates $K$.*

Corollary 2.4, (b) actually characterizes the spaces not containing a copy of $l_1$ among the separable Banach spaces (see [FLP], Theorem 5.14 and the references quoted in this paper).

## 3. Generalization of the notion of boundary

Let $X$ be a separable Banach space and let $W$ be the unit ball of some (equivalent) norm on $X$. We denote by $W^*$ (resp. $W^{**}$) the corresponding unit ball in $X^*$ (resp. $X^{**}$).

*Definition 3.1:* Let $P$ be a subset of $\partial W$. We say that $B \subset \partial W^*$ is a $P$-boundary if, for every $x \in P$, there is an $f \in B$ with $f(x) = 1$.

We shall usually consider subsets $P$ of $\partial W$ of the form $P_{A,W} = A \cap \partial W$ where $A$ is a cone in $X$ with vertex 0 (i.e., $x \in A \Longrightarrow \lambda x \in A$ for every real $\lambda$). The reason we work here with cones $A$ rather than just subsets of $\partial W$ is in order to enable us to pass smoothly from one equivalent norm to another such norm. In most of the results of this section we are forced to work with the class of all norms equivalent to a given norm.

Our main interest here is in the following property:

(*) For any equivalent norm on $X$ and any $P_{A,W}$-boundary $B$ we have that $W^*$ is the norm-closed convex hull of $B$.

Property (*) is clearly a stronger version than the assertion in Corollary 2.4. Therefore, it is not surprising that we will make (implicitly or explicitly) the assumption that either (a) or (b) of Corollary 2.4 holds.

The situation is rather simple if $X$ is reflexive.

PROPOSITION 3.2: *Let $X$ be a reflexive space and let $A \subset X$ be a cone. Then (*) holds if and only if $A$ is dense in $X$.*

*Proof:* Let $X$ be reflexive and $P = A \cap S_X$ be dense in $S_X$. Let $f \in \operatorname{str\,exp} B_{X^*}$ and $x \in S_X$ be such that $f(x) = 1$. Let $\{x_n\}_{n=1}^{\infty}$ be points in $P$ which converge to $x$ and let $\{f_n\}_{n=1}^{\infty} \subset B$ be such that $f_n(x_n) = 1$ for all $n$. Then $f_n(x) \to 1$ and, since $f$ is strongly exposed, $\|f - f_n\| \to 0$. It follows that the norm-closure of $B$ contains $\operatorname{str\,exp} B_{X^*}$. Consequently (see, e.g., [P]), $\operatorname{cl\,co} B = B_{X^*}$.

Conversely, assume that $A$ is not dense in $X$. Let $z \in S_X$ and $\delta > 0$ be such that $A \cap (z + \delta B_X) = \emptyset$. Let $h \in S_{X^*}$ be such that $h(z) = 1$ and let $\gamma$ be the hyperplane $\{x \in X : h(x) = 0\}$. Consider now the cylinder

$$W = \operatorname{co}\{\pm(z + (\delta B_X \cap \gamma))\}.$$

$W$ is clearly the unit ball of an equivalent norm in $X$. We shall show that in this norm (*) fails. Indeed, if $x \in A \cap \partial W$ then $x = \lambda z + y$ with $|\lambda| < 1$ and $y \in \delta B_X \cap \gamma$. Hence for $f \in \partial W^*$ with $f(x) = 1$ we have $f(z) = 0$ (otherwise $f$ would attain a value larger than 1 at $z + y$ or $-z + y$). It follows that for $B = \Sigma(P_{A,W})$ we have $B \subset z^{\perp}$ and, in particular, the closed linear span of $B$ is different from $X^*$. ∎

*Remark:* The "if" part of Proposition 3.2 characterizes reflexive spaces, at least among dual spaces. Indeed, let $X = E^*, A = \Sigma(B_E), P = S_X \cap A$ and $B = S_E \subset E^{**} = X^*$. Then $B$ is a $P$-boundary, $A$ is dense in $X$ (by the Bishop–Phelps theorem) and (*) fails. On the other hand, reflexivity is not applied in the proof of the "only if" part. We will come back to this observation below.

We consider next the following property:

(**) $\operatorname{cl\,co} \Sigma(P_{A,W}) = W^*$ for any equivalent norm on $X$.

Note that in the statement of (**) no specific boundary appears. Obviously (**)$\Longrightarrow$(*). We shall prove below that if (a) or (b) in Corollary 2.4 hold, then (*)$\Longrightarrow$(**) and this is in a sense a generalization of Corollary 2.4. The method of proof will be similar to the arguments used in Section 2. Before we prove this we make some comments on (**). We first note that by the separation theorem (**) is equivalent to:

(***) For any slice

$$S(F, \alpha) = \{f \in W^* : F(f) \geq 1 - \alpha\}, \quad F \in \partial W^{**}, \quad \alpha \in (0, 1)$$

of $W^*$ there is an $f \in S(F, \alpha) \cap \partial W^*$ and an $x \in P_{A,W}$ with $f(x) = 1$.

PROPOSITION 3.3: *If $B_X \setminus A$ is not norming ($B_X$ is the unit ball of some norm in $X$ and the assertion that $B_X \setminus A$ is not norming means that 0 is not an interior point of $\operatorname{cl}\operatorname{co}(B_X \setminus A)$), then $A$ has (**).*

*Proof:* We note first that if $B_X \setminus A$ is not norming, the same is true for $W \setminus A$ for any unit ball of an equivalent norm. If $A$ does not have (**) (or equivalently (***)) there is a slice $S = S(F, \alpha)$ of $W^*$ for some equivalent norm so that, for any $f \in \Sigma(W) \cap \partial W^* \cap S$, there is an $x \in \partial W \setminus A$ with $f(x) = 1$. By the Bishop–Phelps theorem we may assume without loss of generality that $F(g) = 1$ for some $g \in \partial W^*$.

Put $W_1^* = (-(1-\alpha/2)g + S) \cap ((1-\alpha/2)g - S)$ and let $W_1 \subset X$ be the polar of $W_1^*$. $W_1$ defines an equivalent norm on $X$. Since $A$ is a cone (and thus symmetric with respect to 0), for every $f \in \partial W_1^* \cap \Sigma(W_1)$ there is an $x \in \partial W_1 \setminus A$ with $f(x) = 1$. Hence, by the separation theorem and the Bishop–Phelps theorem $\operatorname{cl}\operatorname{co}(\partial W_1 \setminus A) = W_1$ and this contradicts the assumption that $\partial W_1 \setminus A$ is not norming. ∎

*Remark:* Proposition 3.2 shows that, in general, the converse to Proposition 3.3 is false. Indeed, let $X$ be a reflexive space and let $A \subset X$ be a dense cone such that $X \setminus A$ is dense too.

We come now to the proof of the equivalence of (*) and (**) under the assumptions (a) or (b) in Corollary 2.4. The main step in the proof is contained in the following lemma.

LEMMA 3.4: *Let $B \subset S_{X^*}$ be such that $w^* \operatorname{cl}\operatorname{co} B = B_{X^*}$. Assume that $B = \bigcup_{i=1}^{\infty} C_i$, let $\varepsilon > 0$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence with $0 < \varepsilon_i < \varepsilon$ for every $i$ and $\lim_i \varepsilon_i = 0$. Put*

$$V^* = w^*\operatorname{cl}\operatorname{co} \bigcup_{i=1}^{\infty} (1+\varepsilon_i) C_i, \quad V = \{x \in X : \sup\{f(x) : f \in V^*\} \le 1\}.$$

*Then*

(i) $V \subset B_X \subset (1+\varepsilon)V$.

(ii) *For $z \in \partial V \cap \operatorname{int} B_X$, $h \in \partial V^*$ with $h(z) = 1$, there is a finite $l$ such that $h \in \operatorname{co} \bigcup_{i=1}^{l} w^*\operatorname{cl}\operatorname{co}\,(1+\varepsilon_i) C_i$.*

*Proof:* (i) Clearly $V^* \subset (1+\varepsilon) B_{X^*}$ and hence $B_X \subset (1+\varepsilon)V$. To check that $V \subset B_X$ we prove that $B_{X^*} \subset V^*$. Since $w^*\operatorname{cl}\operatorname{co} B = B_{X^*}$ it suffices to show that $0 \in V^*$. Otherwise, we can find by the separation theorem an $x \in X$ and an $a > 0$ so that $(1+\varepsilon_i)f(x) > a$ for every $f \in C_i$, $1 \leq i < \infty$. This, however, contradicts the assumption that $w^*\mathrm{cl\,co}\, B = B_{X^*}$ (and thus that $B$ contains elements $f$ with $f(x) < 0$).

(ii) The proof runs along the lines of the proof of Theorem 2.3 and thus we present only an outline of the proof. Put $B_1 = \bigcup_{i=1}^{\infty}(1+\varepsilon_i)C_i$ and let $h$ and $z$ be as in (ii), in particular $||z|| < 1$. By the Krein–Milman theorem there is a probability measure $\mu$ supported by $w^*\,\mathrm{cl}\, B_1$ representing $h$. Put $\gamma = \{f \in X^* : f(z) = 1\}$. Clearly, $\mathrm{supp}\,\mu \subset w^*\,\mathrm{cl}\, B_1 \cap \gamma$.

Next, we claim that $\gamma$ meets at most finitely many of the sets $w^*\,\mathrm{cl}(1+\varepsilon_i)C_i$. Indeed, otherwise we would get (since $\varepsilon_i \to 0$) that $\gamma$ meets $S_{X^*}$, but this contradicts the fact that $||z|| < 1$. Consequently, $\mu$ is supported on $\bigcup_{i=1}^{l} w^*\,\mathrm{cl}(1+\varepsilon_i)C_i$ for some finite $l$. ∎

THEOREM 3.5: *Assume that $A$ is a cone in $X$ and that (**) holds. Let $P = A \cap S_X$. Then any $P$-boundary $B$ (I)-generates $B_{X^*}$.*

*Proof:* By the "only if" part of the proof of Proposition 3.2, $A$ is dense in $X$ (see the remark following Proposition 3.2). Thus $B$ (W)-generates $B_{X^*}$, i.e., $w^*\,\mathrm{cl}\, B = B_{X^*}$.

Let $B = \bigcup_{i=1}^{\infty} C_i$ and put $D = \mathrm{cl\,co}\bigcup_{i=1}^{\infty} w^*\mathrm{cl\,co}\, C_i$. If $D$ is a proper subset of $B_{X^*}$, then there are an $F \in S_{X^{**}}$ and $\alpha > 0$ such that $\sup\{F(f) : f \in D\} < 1-\alpha$.

Take a sequence $\varepsilon_i, 0 < \varepsilon_i < \alpha/2$ and $\lim_i \varepsilon_i = 0$. Define $V$ and $V^*$ as in Lemma 3.4 and put $D_1 = \mathrm{cl\,co}\bigcup_{i=1}^{\infty} w^*\mathrm{cl\,co}\,(1+\varepsilon_i)C_i$.

Then $\sup\{F(f) : d \in D_1\} < (1-\alpha)(1+\alpha/2) < 1-\alpha/2$. Assume that there are an $f \in \partial V^* \cap S(F, \alpha/2)$ and an $x \in \partial V \cap \int B_X$ with $f(x) = 1$. Then by Lemma 3.4, $f \in D_1$, which is a contradiction. Thus for any $f \in \partial V^* \cap S(F, \alpha/2)$ and $x \in \partial V$ we have $x \in S_X$. Such an $x$ cannot belong to $P$. Indeed, since $B$ is a boundary for $P$ there is a $g \in B$ such that $g(x) = 1$. However, this $g \in C_i$ for some $i$ and $(1+\varepsilon_i)g_i \in V^*$, i.e., $(1+\varepsilon_i)g_i(x) > 1$, which contradicts the assumption that $x \in V$.

Thus we have just proved that whenever $f \in \partial V^* \cap S(F, \alpha/2)$ and $x \in \partial V$ with $f(x) = 1$ we have $x \in S_X \setminus P$. This shows that (***) does not hold for $A$ and this contradicts our assumption. ∎

COROLLARY 3.6: *Let $A$ be a cone in a separable space $X$. Assume that (**) holds. Then for any $S_X \cap A$-boundary $B$ we have $\mathrm{cl\,co}\, B = B_{X^*}$ if either $B$ is separable or $X \not\supset l_1$.*

If $X \setminus A$ is a set of the first category, this does not guarantee that $A$ has (**). Consider the classical quasi-reflexive space $J$ of James. There is an equivalent norm on $J$ such that in it $J^*$ is smooth (see, e.g., [S]). Taking this norm in $J$ let $X = J^*$, $A = \Sigma(B_J) \subset J^*$ and $P = A \cap S_{J^*}$. Since $J$ has the RNP it follows from (ii) in the introduction that $X \setminus A$ is of the first category. Since the norm in $X$ is smooth, $\Sigma(P) \subset J \subset J^{**} = X^*$. Therefore $\mathrm{cl}\,\mathrm{co}(S_{X^*} \cap \Sigma(P)) \subset B_J$, which is a proper subset of $B_{X^*} = B_{J^{**}}$. Thus $A$ fails to have (**).

There is, however, a thin subclass of the class of sets of the first category so that if $X \setminus A$ belongs to this thinner subclass then $A$ always has (**) if $X \not\supset l_1$. This subclass is the class of all proper operator ranges. Recall that a subspace $M$ (not closed in general) of $X$ is called a proper operator range if there are a Banach space $Y$ and a bounded linear operator $T\colon Y \to X$ such that $M = T(Y) \neq X$.

THEOREM 3.7: *Let $X$ be a separable Banach space with $X \not\supset l_1$. Let $M \subset X$ be a proper operator range. Then $A = X \setminus M$ satisfies (*)*

For the proof of this theorem we need 4 lemmas. The first two are standard and we omit their proofs.

LEMMA 3.8: *Let $M \subset X$ be a proper operator range. Then either $M$ is a closed subspace of finite codimension or* $\mathrm{codim}\, M = \infty$.

LEMMA 3.9: *Let $T\colon Y \to X$ be a bounded linear operator from a Banach space $Y$ into $X$. The following statements are equivalent:*

(i) $\mathrm{codim}\, T(Y) = \infty$.

(ii) *For every subspace $L$ of $X^*$ of finite codimension and every $\varepsilon > 0$ there is an $f \in S_L$ such that $\|T^* f\| \leq \varepsilon$.*

LEMMA 3.10 ([F]): *Let $A\colon E \to X$ be a bounded linear operator from a Banach space $E$ into $X$ with* $\mathrm{codim}\, A(E) = \infty$. *Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $X$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers. Then there are sequences $\{y_i\}_{i=1}^{\infty}$ in $X$ and positive numbers $\{\gamma_i\}_{i=1}^{\infty}$ such that $\|x_i - y_i\| < \varepsilon_i$ for every $i$ and* $\mathrm{cl}\,\mathrm{co}\{\pm\gamma_i y_i\}_{i=1}^{\infty} \cap A(E) = \{0\}$.

LEMMA 3.11: *Let $T\colon Y \to X$ be a bounded linear operator from a Banach space $Y$ into a Banach space $X$ such that* $\mathrm{codim}\, T(Y) = \infty$. *Let $\{z_i\}_{i=1}^{\infty}$ be a normalized basic sequence in $X$ that has a $w^*$-limit point $F \in X^{**} \setminus X$. Then there are a subsequence $\{z_{i_k}\}_{k=1}^{\infty}$ of $\{z_i\}_{i=1}^{\infty}$ and a basic sequence $\{u_k\}_{k=1}^{\infty}$ in $X$ such that $\lim_k \|z_{i_k} - u_k\| = 0$ and the closed linear span $[u_k]_{k=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ satisfies $[u_k]_{k=1}^{\infty} \cap T(Y) = \{0\}$.*

*Proof:* Put $L_1 = \operatorname{Ker} F \subset X^*$. By using Lemma 3.9 find an $f_1 \in S_{L_1}$ with $||T^* f_1|| < 2^{-2}$. Since 0 is a limit point of the sequence $\{f_1(z_i)\}_{i=1}^\infty$ it follows that there is a $z_{i_1}$ with $|f_1(z_{i_1})| < 2^{-2}$. Clearly,

$$\sup\{f_1(tz_{i_1} + Ty) : t \in [-1, 1], y \in B_Y\} < 2^{-1}.$$

Let $v_1 \in B_X$ be such that $f_1(v_1) > 1 - 2^{-1}$. Put $L_2 = L_1 \cap [z_{i_1}, v_1]^\perp$. Choose $f_2 \in S_{L_2}$ with $||T^* f_2|| < 2^{-3}$ and, by using that $F$ is a $w^*$-limit point of $\{z_i\}_{i=1}^\infty$ and $f_1, f_2 \in \operatorname{Ker} F$, find a $z_{i_2}, i_2 > i_1$, with $|f_k(z_{i_2})| < 2^{-3}, k = 1, 2$. We get

$$\sup\{f_k(t_1 z_{i_1} + t_2 z_{i_2} + Ty) : t_1, t_2 \in [-1, 1], y \in B_Y\} < 2^{-k}, \quad k = 1, 2.$$

Let $\{v_l\}_{l=1}^{n_2} \subset B_X$ be such that $\{v_l|_{[f_1, f_2]}\}_{l=1}^{n_2}$ is a $2^{-2}$-net in $B_{[f_1, f_2]^*}$. Put $L_3 = L_1 \cap [z_{i_k}, v_l]^\perp_{k=1,2;l=1,\ldots,n_2}$ and choose $f_3 \in S_{L_3}$ such that $||T^* f_3|| < 2^{-4}$. By using that $F$ is a $w^*$-limit point of $\{z_i\}_{i=1}^\infty$ and $f_1, f_2, f_3 \in \operatorname{Ker} F$, find a $z_{i_3}, i_3 > i_2$, with $|f_k(z_{i_3})| < 2^{-4}, k = 1, 2, 3$. We get

$$\sup\{f_k(t_1 z_{i_1} + t_2 z_{i_2} + t_3 z_{i_3} + Ty) : t_1, t_2, t_3 \in [-1, 1], y \in B_Y\} < 2^{-k}, k = 1, 2, 3.$$

Let $\{v_l\}_{l=1}^{n_3} \subset B_X$ be such that $\{v_l|_{[f_1, f_2, f_3]}\}_{l=1}^{n_3}$ is a $2^{-3}$-net in $B_{[f_1, f_2, f_3]^*}$. It is clear how to continue this construction. In this way we get a basic sequence $\{f_k\}_{k=1}^\infty \subset S_{X^*}$ and a subsequence $\{z_{i_k}\}_{k=1}^\infty$ such that

$$\sup\left\{f_m\left(\sum_{k=1}^n t_k z_{i_k} + Ty\right) : t_k \in [-1, 1], y \in B_Y\right\} < 2^{-m} \tag{3.1}$$

holds for any $n$ and $m$.

We show that $\operatorname{codim}([z_{i_k}]_{k=1}^\infty + T(Y)) = \infty$. Put $E = [z_{i_k}]_{k=1}^\infty \oplus_\infty Y$ and define $A: E \to X$ as follows:

$$A(x + y) = x + Ty, \quad x \in [z_{i_k}]_{k=1}^\infty, \quad y \in Y.$$

In view of Lemma 3.9 we have to check that for any subspace $L \subset X^*$ of finite codimension and for any $\varepsilon > 0$ there is an $f \in S_L$ with $||A^* f|| < \varepsilon$, i.e.,

$$\sup\left\{f\left(\sum_{k=1}^\infty t_k z_{i_k} + Ty\right) : \sum_{k=1}^\infty t_k z_{i_k} \in B_{[z_{i_k}]_{k=1}^\infty}, y \in B_Y\right\} < \varepsilon.$$

Let $a$ and $b$ be the basis constants of the bases $\{f_k\}$ and $\{z_{i_k}\}$ respectively. Since

$\operatorname{codim} L = \infty$ it follows that for any $m$ there is an $f = \sum_{n=m}^{\infty} c_n f_n \in S_L$. Write

$$\begin{aligned}\left|f\Big(\sum_{k=1}^{\infty} t_k z_{i_k} + Ty\Big)\right| \le & \left|\Big(\sum_{n=m}^{\infty} c_n f_n\Big)\Big(\sum_{k=1}^{\infty} t_k z_{i_k}\Big)\right| + \left|\Big(\sum_{n=m}^{\infty} c_n f_n\Big)(Ty)\right| \\ \le & 4ab \sum_{n=m}^{\infty} |c_n/2a| \left| f_n\Big(\sum_{k=1}^{\infty} (t_k/2b) z_{i_k}\Big)\right| \\ & + 2a \sum_{n=m}^{\infty} |c_n/2a| \|T^* f_n\| \\ \le & 4ab \sum_{n=m}^{\infty} 2^{-n} + 2a \sum_{n=m}^{\infty} 2^{-n-1} \le 4ab2^{-m+1} + 2a2^{-m}.\end{aligned}$$

By taking $m$ large enough we complete the proof that $\operatorname{codim}([z_{i_k}]_{k=1}^{\infty} + T(Y)) = \operatorname{codim} A(E) = \infty$.

Take any basic sequence $\{x_k\} \subset S_X$ and let $\{\varepsilon_k\}$ be a stability sequence of this basic sequence. Apply Lemma 3.10 and find sequences $\{\gamma_k\}$ and $\{y_k\}$ such that $\|x_k - y_k\| < \varepsilon_k, k = 1, 2, \ldots$ and

$$\operatorname{cl} \operatorname{co}\{\pm \gamma_k y_k\}_{k=1}^{\infty} \cap A(E) = \{0\}. \tag{3.2}$$

In particular, $\{y_k\}$ is a basic sequence. Without loss of generality we may assume that $\{\gamma_k\}$ is a stability sequence of a basic sequence $\{z_{i_k}\}$ and $\sum_{k=1}^{\infty} \gamma_k < \infty$. Put $u_k = z_{i_k} + \gamma_k y_k, k = 1, 2, \ldots$. Clearly, $\{u_k\} \subset X$ is a basic sequence with $\lim \|z_{i_k} - u_k\| = 0$. Check that $[u_k] \cap T(Y) = \{0\}$. Let $u = \sum a_k u_k \in T(Y)$. We have $u = \sum a_k u_k = \sum a_k z_{i_k} + \sum a_k \gamma_k y_k \in T(Y)$, and hence $\sum a_k \gamma_k y_k = u - \sum a_k z_{i_k} \in A(E)$. By (3.2), $\sum a_k \gamma_k y_k = 0$. Since $\{y_k\}$ is a basic sequence it follows that $a_k = 0$, $k = 1, 2, \ldots$, i.e., $u = 0$ and the proof is complete. ∎

*Remark:* The condition that $\{z_i\}_{i=1}^{\infty}$ has a $w^*$-limit point $F \in X^{**} \setminus X$ is not essential. Lemma 3.11 remains true without this condition. Indeed, if a basic sequence $\{z_i\} \subset X$ does not have $w^*$-limit points in $X^{**} \setminus X$, then it $w$-converges to 0 and the same proof works.

*Proof of Theorem 3.8:* Let $T\colon Y \to X$ be a bounded linear operator from a Banach space $Y$ into $X$ with $T(Y) \neq X$. Put $A = X \setminus T(Y)$, and consider two cases according to Lemma 3.8.

Case 1: $\operatorname{codim} T(Y) < \infty$, and $T(Y)$ is a proper closed subspace of $X$. Then $B_X \cap T(Y)$ is not norming. By Proposition 3.3 the set $A$ has (**), and thus also (*) by Corollary 3.6.

CASE 2: $\operatorname{codim} T(Y) = \infty$. Assume to the contrary that there is a $B \subset S_{X^*}$ such that, for any $x \in X \setminus T(Y)$, there is an $f \in B$ with $f(x) = ||x||$ but $\operatorname{cl} \operatorname{co} B \neq B_{X^*}$. Let $F \in S_{X^{**}}$ be such that $\sup\{F(f) : f \in B\} \leq 1 - \alpha, \alpha > 0$. Since $X$ is separable and does not contain $l_1$ it follows from the Odell–Rosenthal theorem that there is a sequence $\{z_i\}_{i=1}^{\infty} \subset S_X$ with $w^* \lim_i z_i = F$. It is well-known that such a sequence contains a basic subsequence. Without loss of generality we may assume that $\{z_i\}_{i=1}^{\infty}$ is basic. By Lemma 3.11 there is a sequence $\{u_k\}_{k=1}^{\infty}$ in $X$ such that $[u_k] \cap T(Y) = \{0\}$ and $\lim_k ||u_k - z_k|| = 0$. Clearly, $F = w^* \lim_k u_k$. Put $E = [u_k]$; then $B|_E$ is a boundary for $B_{E^*}$. By Corollary 2.4, $\operatorname{cl} \operatorname{co} B|_E = B_{E^*}$. Since $F \in w^* \operatorname{cl} E = E^{**}$ and $||F|| = 1$ we deduce that

$$\sup\{F(g) : g \in B\} = \sup\{F(g) : g \in B|_E\} = \sup\{F(g) : g \in B_{E^*}\} = 1.$$

This, however, contradicts the choice of $F$. ∎

## 4. On the set of functionals that do not attain their norms

In this section we prove several results on the size of the set $X^* \setminus \Sigma(B_X)$ for non-reflexive $X$. We use here the technique developed in Section 3.

THEOREM 4.1: *Let $X$ be a separable non-reflexive Banach space with unit ball $B_X$. Let $a \in (0,1)$, $\varepsilon > 0$ with $a(1+2\varepsilon) < 1$, let $0 < \varepsilon_i < \varepsilon$ for all $i$ with $\lim_i \varepsilon_i = 0$ and let $\{x_i\}_{i=1}^{\infty} \subset X$. Put $D_i = S_X \cap (x_i + aB_X), i = 1, 2, \ldots$. Assume that $E$ is a closed subspace of $X^*$ with $\phi(X) \neq E^*$, where $\phi$: $X^{**} \to X^{**}/E^{\perp} = E^*$ is the natural quotient map. Then there exists an $h \in S_E$ with*

$$\sup\{h(y) : y \in D_i\} \leq 1 - \varepsilon_i, \quad i = 1, 2, \ldots.$$

*Proof:* Without loss of generality we may assume that $||x_i|| \leq 2$ for all $i$ and that $\{x_i\}_{i=1}^{\infty}$ is dense in $2B_X$. Assume to the contrary that for any $h \in S_E$ there is an $i$ such that $\sup\{h(y) : y \in D_i\} > 1 - \varepsilon_i$.

Take $\beta \in (0, 1)$ so that

$$r = (1 + 2\varepsilon)(a + 2(1 - \beta))/\beta < 1, \tag{4.1}$$

and put

$$C_i = \{\phi(x)/||\phi(x)|| : x \in D_i, ||\phi(x)|| > \beta\} \tag{4.2}$$

and $B = \bigcup_{i=1}^{\infty} C_i$ (we consider $X$ in a natural way as a subspace of $X^{**}$ so $\phi$ is defined also on $X$). For any $e \in S_E$ there is an $x \in S_X$ with $e(x) = \phi(x)(e)$ and

$||\phi(x)||$ as close to 1 as we wish. It follows that $w^*\mathrm{cl\,co}\, B = B_{E^*}$. Fix an $i$ and $z \in C_i$, where $z = \phi(x)/||\phi(x)||$, $x \in D_i$, $||\phi(x)|| \geq \beta$. We have

$$||z - \phi(x_i)|| = \left\| \frac{\phi(x)}{||\phi(x)||} - \frac{\phi(x_i)}{||\phi(x)||} + \frac{\phi(x_i)}{||\phi(x)||} - \phi(x_i) \right\| \leq a/\beta + 2(1-\beta)/\beta.$$

Hence by (4.1)

$$w^*\,\mathrm{cl\,co}\, C_i \subset \phi(x_i) + \frac{r}{1+2\varepsilon} B_{E^*} \subset \phi(x_i) + \frac{r}{1+2\varepsilon_i} B_{E^*}. \tag{4.3}$$

Define now the sets $V^*$ and $V$ as in Lemma 3.4, i.e., put

$$V^* = w^*\,\mathrm{cl\,co} \bigcup_{i=1}^{\infty} (1+2\varepsilon_i) C_i, \quad V = \{x \in E : \sup\{f(x) : f \in V^*\} \leq 1\}.$$

By Lemma 3.4, (i), $V \subset B_E \subset (1+2\varepsilon)V$.

Next we claim that $V \subset \mathrm{int}\, B_E$. Indeed, if $h \in S_E$ then by assumption we have, for some $i$, $\sup\{h(y) : y \in D_i\} > 1 - \varepsilon_i$. Hence

$$\sup\{h(x) : x \in (1+2\varepsilon_i)C_i\} \geq \sup\{h(y) : y \in (1+2\varepsilon_i)D_i\} > (1+2\varepsilon_i)(1-\varepsilon_i) > 1,$$

and thus $h \notin V$.

By Lemma 3.4, (ii), we deduce that for $g \in \Sigma(V) \cap \partial V^*$ there is a finite $l$ so that

$$g \in \mathrm{co} \bigcup_{i=1}^{l} w^*\,\mathrm{cl\,co}(1+2\varepsilon_i)C_i,$$

and thus by (4.3)

$$d(g, \phi(X)) \leq r < 1.$$

Since $\mathrm{cl}\,\phi(X)$ is a proper subspace of $E^*$ it follows from the Riesz lemma that there is a $k \in S_{E^*}$ with $d(k, \phi(X)) > r$. We have that $V^* \supset B_{E^*}$, thus for some $\lambda \geq 1$, $\lambda k \in \partial V^*$. By the Bishop–Phelps theorem there is a $g \in \Sigma(V) \cap \partial V^*$ which is close to $\lambda k$ so that $d(g, \phi(X)) > r$. We have thus arrived at a contradiction. ∎

*Remark:* The assumption in the statement of Theorem 4.1 clearly holds if we take $E = X^*$. The condition $\phi(X) \neq E^*$ is essential. Put $X = Z^*$ for some Banach space $Z$ and $E = Z \subset Z^{**} = X^*$. Then $E \subset \Sigma(B_X)$.

In the previous theorem we dealt with balls of a "large" (close to 1) radius. In our next result we consider balls with a small radius.

PROPOSITION 4.2: *Let $Y$ be a separable non-reflexive Banach space. Then for every $\varepsilon > 0$ there exists a closed convex body $V_* \subset Y$ with $\frac{1}{1+\varepsilon}V_* \subset B_Y \subset V_*$ (with polar set $V$ in $Y^*$ and bipolar set $V^*$ in $Y^{**}$) such that, for every $F \in \partial V^* \cap \Sigma(V)$ with $d(F,Y) > \varepsilon$, we have*

$$\sigma(F) = \{f \in \partial V : F(f) = 1\} \subset Y^* \setminus \Sigma(B_Y).$$

*Proof:* Use Lemma 3.4 for $X = Y^*, B = S_Y (\subset X^* = Y^{**})$ and $C_i = (y_i + \frac{\varepsilon}{2}B_Y) \cap S_Y$ where $\{y_i\}$ is a dense subset of $S_Y$. Define $V$ and $V^*$ as in Lemma 3.4 and put $V_* = V^* \cap Y$. Clearly, $V$ and $V^*$ are the polar (resp. bipolar) sets of $V_*$. By Lemma 3.4 we have $V \subset B_Y \subset (1+\varepsilon)V$ and thus $\frac{1}{1+\varepsilon}V_* \subset B_Y \subset V_*$. Let $f \in \partial V$ and $x \in S_Y$ be such that $f(x) = \|x\|$. Since $S_Y = \bigcup_{i=1}^{\infty} C_i$ it follows that $x \in C_j$ for some $j$. Hence $(1+\varepsilon_j)x \in V_*$ and, as $f \in V$, we have $f((1+\varepsilon_j)x) \le 1$ and therefore $\|f\| < 1$. In other words $\partial V \cap \Sigma(B_Y) \subset \operatorname{int} B_Y$. By Lemma 3.4 (ii) we get that, for every $F \in \Sigma(V) \cap \partial V^*$ with $\sigma(F) \cap \Sigma(B_Y) = \emptyset$, we have for some finite $l$

$$F \in \operatorname{co} \bigcup_{i=1}^{l} w^* \operatorname{cl} \operatorname{co} (1+\varepsilon_i) C_i$$

and thus $d(F,Y) < \varepsilon$. ∎

Our next result, which uses again the notion of a proper operator range, shows that in every separable non-reflexive space the set of functionals which do not attain their maximum on the unit ball cannot be too thin.

THEOREM 4.3: *Let $X$ be a non-reflexive separable space. Then the set of bounded linear functionals on $X$ which do not attain their norm on $B_X$ is not a subset of a proper operator range.*

*Proof:* Assume to the contrary that there is a bounded linear operator $T: Y \to X^*$ such that $T(Y) \neq X^*$ and each $f \in X^* \setminus T(Y)$ attains its norm on $B_X$. Consider two cases.

CASE 1: $\operatorname{codim} T(Y) < \infty$. Then $T(Y)$ is a proper closed subspace of $X^*$ and thus $B_{X^*} \cap T(Y)$ is not norming for $X^{**}$. By Proposition 3.3 the cone $A = X^* \setminus T(Y)$ has (**). Put $B = S_X \subset X^{**}$. By our assumption $B$ is a (separable) $B_{X^*} \setminus T(Y)$ boundary. Hence, by Corollary 3.6, $\operatorname{cl}\operatorname{co} B = B_{X^{**}}$, which is impossible.

CASE 2: $\operatorname{codim} T(Y) = \infty$. Since $X$ is separable and non-reflexive, $B_{X^*}$ contains a sequence $\{f_n\}$ which $w^*$-converges to 0 but does not have a $w$-limit point. Without loss of generality we can assume that $\{f_n\}$ is a basic sequence. Let

$F \in X^{***} \setminus X^*$ be a $w^*$-limit point of $\{f_n\}$. By Lemma 3.11 there are a sequence $\{u_k\} \subset X^*$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ so that $\lim_k \|u_k - f_{n_k}\| = 0$ and $[u_k] \cap T(Y) = \{0\}$. Let $G \in X^{***} \setminus X^*$ be a $w^*$-limit point of $\{u_k\}$. Since $\{u_k\}$ tends $w^*$ to 0 it follows that $G \in X^{\perp}$. Since $[u_k] \cap T(Y) = \{0\}$ it follows from our assumption that $S_X|_{[u_k]}$ (considered as a subset of $[u_k]^*$) is a separable boundary for $[u_k]$. By Corollary 3.6, $B_{[u_k]^*}$ is the closed convex hull of $S_X|_{[u_k]}$. However, $G \in w^*\,\mathrm{cl}[u_k] = [u_k]^{**}, G \neq 0$ and $G|_X = 0$. We have thus arrived at a contradiction. ∎

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